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Stability and asymptoticity of Volterra difference equations: A progress report

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ABSTRACT

We survey some of the fundamental results on the stability and asymptoticity of linear Volterra difference equations. The method of Z-transform is heavily utilized in equations of convolution type. An example is given to show that uniform asymptotic stability does not necessarily imply exponential stabilty. It is shown that the two notions are equivalent if the kernel decays exponentially. For equations of nonconvolution type, Liapunov functions are used to find explicit criteria for stability. Moreover, the resolvent matrix is defined to produce a variation of constants formula. The study of asymptotic equivalence for difference equations with infinite delay is carried out in Section 6. Finally, we state some problems.

0. Introduction

Burton [3] gave a comprehensive exposition on the stability of Volterra integrodifferential and integral equations. Brunner and Van der Houwen [2] provided numerical methods to solve Volterra equations. It is well known, [2,1,17], that numerical methods applied to Volterra equations lead to Volterra difference equations. A systematic study of Volterra difference equations may be traced to papers by the author that appeared in 1993 [11], 1994 [12], 1996 [10] and 2003 [9]. Independently, Crisci et al. developed a parallel theory [5–8]. Interesting results on the stability and boundedness of solutions of Volterra difference equations may be found in [4,18]. Readable accounts on Volterra difference equations and the Z-transform may be found in [13]. The main objective of this paper is to present the latest developments in the theory of linear Volterra difference equations of both convolution and nonconvolution types. It is not a survey of all the work done but rather a more focused report on the work of the author and his collaborators.

1. Scalar linear equations of convolution type

Consider the equation

$$x(n+1) = ax(n) + \sum_{j=0}^{n} b(n-j)x(j), \tag{1.1}$$

where $n \in \mathbb{Z}^+$, $a \in \mathbb{R}$, and $b(n) : \mathbb{Z}^+ \to \mathbb{R}$ are given sequences. This equation may be considered as the discrete analogue of the famous Volterra integrodifferential equation

$$x'(t) = ux(t) + \int_0^t v(t-s)x(s)\,\mathrm{d}s.$$

One of the most effective methods of dealing with Eq. (1.1) is the Z-transform method which we will review.

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Definition 1.1 ([13]). The Z-transform $Z(x(n))$ or $\tilde{x}(z)$ of a sequence $x(n)$, $n \in \mathbb{Z}^+$ ($x(n) = 0$ for $n < 0$) is defined by $\tilde{x}(z) = Z(x(n)) = \sum_{j=0}^{\infty} x(j)z^{-j} * Z(x(n+k)) = z^k\tilde{x}(z) - \sum_{r=0}^{k-1} x(r)z^{k-r}$. The convolution of two sequences $x(n)$ and $y(n)$ is defined by

$$x(n) * y(n) = \sum_{j=0}^{n} x(n-j)y(j) = \sum_{j=0}^{n} x(n)y(n-j)$$

$$\boxed{Z(x(n) * y(n)) = \tilde{x}(z) \cdot \tilde{y}(z)}$$

Eq. (1.1) may be written as

$$x(n+1) = ax(n) + b(n) * x(n).$$

Taking the Z-transform of both sides yields

$$\tilde{x}(z) = \frac{zx(0)}{z - a - \tilde{b}(z)} \tag{1.2}$$

or

$$\tilde{x}(z) = zx(0)g^{-1}(z), \tag{1.2$'$}$$

where

$$g(z) = z - a - \tilde{b}(z). \tag{1.3}$$

Lemma 1.2 ([12,13]). *The zeros of $g(z)$ all lie in the region $|z| < c$ for some real positive constant c. Moreover, $g(z)$ has finitely many zeros z with $|z| \geq 1$, provided that $x(n) \in \ell^1$ (summable $\sum_{i=0}^{\infty} |x(i)| = \|x\|_1 < \infty$).*

Proof. Suppose that all the zeros of $g(z)$ do not lie in any region $|z| < c$ for any $c > 0$. Then there exists a sequence of zeros $\{z_i\}$ of $g(z)$ with $z_i \to \infty$ as $i \to \infty$. Now

$$|z_i - a| = |\tilde{b}(z_i)| \leq \sum_{n=0}^{\infty} |b(n)||z_i|^{-n}. \tag{1.4}$$

Note that the left hand side of Eq. (1.4) tends to ∞ as $i \to \infty$, while the right hand side tends to $b(0)$ (by inspection), which is a contradiction. This proves the first part of the lemma.

Since $x(n) \in \ell^1$, the "radius" of convergence of $\tilde{x}(z)$ is $R = 1$. Hence $\tilde{x}(z)$ can be differentiated term by term in its region of convergence $|z| > 1$. Thus $\tilde{x}(z)$ is analytic in the region $|z| > 1$. Furthermore, since $x(n) \in \ell^1$, it follows that $\tilde{x}(z)$ is analytic on $|z| \geq 1$. Hence $\tilde{x}(z)$ is analytic in the region $1 \leq |z| \leq c$ and consequently $g(z)$ has finitely many zeros for $|z| \geq 1$. $\square$

We now utilize this lemma to provide conditions for uniform stability and uniform asymptotic stability of the zero solution of Eq. (1.1).

Since $\tilde{x}(z) = x(0)zg^{-1}(z)$, it follows that

$$x(n) = \frac{1}{2\pi i} \oint_{\gamma} x(0)z^n g^{-1}(z)\, dz,$$ [1]

where γ is the origin-centred circle that includes all the zeros of $g(z)$. By the residue theorem

$$x(n) = x(0) \cdot \text{sum of residues of } z^n g^{-1}(z). \tag{1.5}$$

Let z_r be a zero of $g(z)$ of order k. Then the Laurent's series expansion $g^{-1}(z) = \sum_{n=-k}^{\infty} g_n(z - z_r)^n$, for some sequence $\{g_n\}$. Now $z^n = [z_r - (z_r - z)]^n = \sum_{i=0}^{n} \binom{n}{i} z_r^{n-i}(z - z_r)^i$. Let K_r be the residue of $x(0)z^n g^{-1}(z)$ at z_r. Then $K_r = x(0)$ the coefficient of $(z - z_r)^{-1}$ in $g^{-1}(z)z^n$. Note that the coefficient of $(z - z_r)^{-1}$ in $g^{-1}(z)z^n$ is given by

$$g_{-k}\binom{n}{k-1} z_r^{n-k+1} + g_{-k+1}\binom{n}{k-r} z_r^{n-k+2} + \cdots + g_{-1}\binom{n}{0} z_r^n.$$

Hence

$$\boxed{x(n) = \sum p_r(n) z_r^n} \tag{1.6}$$

This formula has important stability results. For the convenience of the reader we now present the relevant stability definitions.

We say that $x(n) = x(n, n_0, \phi)$ is a solution of Eq. (1.1) with a bounded initial function $\phi : [0; n_0] \to \mathbb{R}^k$ if it satisfies Eq. (1.1) for $n \geq n_0$ and $x(j) = \phi(j)$ for $n < n_0$

[1] Cauchy Integral Formula $x(n) = \frac{1}{2\pi i} \oint_{\gamma} \tilde{x}(z)z^{n-1}\, dz$, where γ is the origin-centred circle that encloses all the poles of $\tilde{x}(z)z^{n-1}$. By the residue theorem $x(n) =$ sum of residues K_i of $\tilde{x}(z)z^{n-1}$. If $\tilde{x}(z)z^{n-1} = \frac{h(z)}{g(z)}$, then $K_i = \lim_{z \to z_i}\left[(z - z_i)\frac{h(z)}{g(z)}\right]$, residue K_i at a simple zero z_i of $g(z)$; $K_i = \frac{1}{(r-1)!} \lim_{z \to z_i} \frac{d^{r-1}}{dz^{r-1}}\left[(z - z_i)^r \frac{h(z)}{g(z)}\right]$ if z_i is a multiple zero of $g(z)$ of order r.

Definition 1.3 ([11]). The zero solution of Eq. (1.1) is stable if for each $\epsilon > 0$ and $n_0 \geq 0$, there is $\delta > 0$ such that $\phi : [0; n_0] \to \mathbb{R}^k$ with $|\phi(n)| < \delta$ on $[0, n_0]$ implies $|x(n, n_0, \phi)| < \epsilon$. It is uniformly stable if ϵ is independent of n_0.

Definition 1.4 ([11]). The zero solution of Eq. (1.1) is uniformly asymptotically stable if it is uniformly stable and there exists $\mu > 0$ such that $x(n, n_0, \phi) \to 0$ as $n \to \infty$, whenever $|\phi(n)| < \mu$ on $[0, n_0]$

Theorem 1.5 ([12,13]). *The zero solution of Eq.* (1.1) *is uniformly stable if and only if the following statements hold.*

(a) $z - a - \tilde{b}(z) \neq 0$ *for all* $|z| > 1$, *and*
(b) *if* z_r *is a zero of* $g(z)$ *with* $|z_r| = 1$, *then the residue of* $z^n g^{-1}(z)$ *at* z_r *is bounded as* $n \to \infty$ *(i.e., the zeros of* $g(z)$ *with* $|z| = 1$ *are simple).*

Proof. If condition (a) holds, then all the zeros of $g(z)$ lie inside the disc $|z| \leq 1$. If $|z_r| < 1$, then its contribution to the solution $x(n)$ is bounded. Now if $|z_r| = 1$ at which, by condition (b), the residue of $x(0)z^n g^{-1}(z)$ is bounded as $n \to \infty$,then, by formula (1.6), its contribution to the solution $x(n)$ is bounded. Hence $|x(n)| \leq L|x(0)|$, for some $L > 0$, and consequently, the zero solution is uniformly stable.

The converse will be omitted. $\square$

Theorem 1.6 ([12,13]). *The zero solution of Eq.* (1.1) *is uniformly asymptotically stable if and only if*

$$\boxed{z - a - \tilde{b}(z) \neq 0 \quad \text{for all } |z| \geq 1}$$

2. Explicit criteria for stability of scalar equations

We start our exposition by establishing a sufficient condition for asymptotic stability.

Theorem 2.1 ([12,13]). *Suppose that* $b(n)$ *does not change sign for* $n \in \mathbb{Z}^+$. *Then the zero solution of Eq.* (1.1) *is asymptotically stable if*

$$|a| + \left|\sum_{n=0}^{\infty} b(n)\right| < 1. \tag{2.1}$$

Proof. Suppose that $b(n) \geq 0$ for $n \in \mathbb{Z}^+$. Let $\beta = \sum_{n=0}^{\infty} b(n)$ and $c(n) = \beta^{-1}b(n)$. Then $\sum_{n=0}^{\infty} c(n) = 1$. Furthermore $|\tilde{c}(z)| \leq \sum_{n=0}^{\infty} |c(n)||z^{-n}| = \sum_{n=0}^{\infty} c(n)|z^{-n}| \leq 1$ for $|z| \geq 1$. Moreover, $\tilde{c}(1) = 1$. Let us write our $g(z)$ in the form $g(z) = z - a - \beta\tilde{c}(z)$. To show uniform stability, we use Theorem 1.6. So assume that there exists a zero z_r of $g(z)$ with $|z_r| \geq 1$. Then $0 = g(z_r) = z_r - a - \beta\tilde{c}(z_r)$. Hence $|z_r - a| = |\beta\tilde{c}(z_r)| \leq |\beta|$. This implies that $|z_r| \leq |a| + |\beta| < 1$, a contradiction. This completes the proof. $\square$

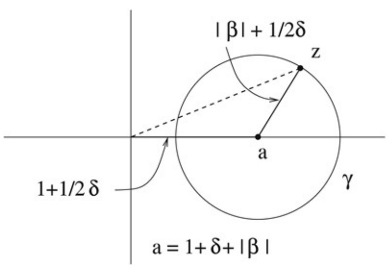
It is still an open question of whether or not condition (2.1) is also a necessary condition for asymptotic stability. Nevertheless, we are able to prove the following partial converse of Theorem 2.1.

Theorem 2.2 ([12,13]). *Suppose that* $b(n)$ *does not change sign on* $\mathbb{Z}^+$. *Then the zero solution of Eq.* (1.1) *is not asymptotically stable if any one of the following conditions holds.*

(i) $a + \sum_{n=0}^{\infty} b(n) \geq 1$,
(ii) $a + \sum_{n=0}^{\infty} b(n) \leq -1$ *and* $b(n) > 0$, *for some* $n \in \mathbb{Z}^+$,
(iii) $a + \sum_{n=0}^{\infty} b(n) \leq -1$ *and* $b(n) < 0$, *for some* $n \in \mathbb{Z}^+$ *and* $\sum_{n=0}^{\infty} b(n)$ *is sufficiently small.*

Proof. We will prove (i). Let $\beta = \sum_{n=0}^{\infty} b(n)$, $c(n) = \beta^{-1}b(n)$. If $a + \beta = 1$, then $g(1) = 1 - a - \beta\tilde{c}(1) = 1 - a - \beta = 0$. Hence by Theorem 2.1, the zero solution of Eq. (1.1) is not asymptotically stable. On the other hand if $a + \beta > 1$, say $a + \beta = 1 + \delta$, for some $\delta > 0$, then we have two separate cases to consider.

(a) If $\beta < 0$, we let γ be the circle in the complex plane with centre at a and radius equal to $|\beta| + \frac{1}{2}\delta$. Then on γ, $|z| > 1$. Hence $|\beta\tilde{c}(z)| \leq |\beta| < |z - a|$.



Let $h(z) = -\beta\tilde{c}(z), f(z) = z - a$. Then on γ, $|h(z)| < |f(z)|$.

[Rouche's Theorem: Suppose that the functions $f(z)$ and $g(z)$ are analytic inside and on a simple closed contour γ in the complex domain, and $|g(z)| < |f(z)|$ at $z \in \gamma$. Then $f(z)$ and $f(z) + g(z)$ have the same number of zeros, counting multiplicities, inside γ.]

Now by Rouche's Theorem $f(z)$ and $g(z) = h(z) + f(z) = z - a - \beta\tilde{c}(z)$ have the same number of zeros inside the circle γ, namely one $z = a$. Thus $g(z)$ has only one zero z_r inside γ with $|z_r| > 1$. Again by using Theorem 2.1, the zero solution of Eq. (1.1) is not asymptotically stable.

(b) Suppose that $\beta > 0$. Since $a + \beta > 1$, $g(z) = 1 - a - \beta < 0$. Moreover, $|\tilde{c}(a + \beta)| = \left|\sum_{n=0}^{\infty} \beta b(n) z^{-n}\right| \leq 1$. Thus $g(a + \beta) = \beta[1 - \tilde{c}(a + \beta)] \geq 0$. Hence g has a zero between 1 and $a + \beta$. By virture of Theorem 2.1, the zero solution of Eq. (1.1) is not asymptotically stable. □

3. Systems of linear Volterra difference equations of convolution type

Consider the k-dimensional system

$$x(n) = Ax(n) + \sum_{j=0}^{n} B(n - j)x(j), \tag{3.1}$$

where $A(a_{ij})$ is a $k \times k$ (real or complex) matrix, $B(n)$ is a sequence of $k \times k$ matrices defined on $\mathbb{Z}^+$. It is always assumed that $B(n) \in \ell_1$ (i.e., $\sum_{j=0}^{n} |B(j)| < \infty$).

Taking the Z-transform of both sides of Eq. (3.1) yields

$$z\tilde{x}(z) - zx(0) = A\tilde{x}(z) + \tilde{B}(z)\tilde{x}(z)$$

or

$$\tilde{x}(z) = zG^{-1}(z)x(0), \tag{3.2}$$

where

$$G(z) = zI - A - \tilde{B}(z). \tag{3.3}$$

In order to provide a more comprehensive characterization of uniform asymptotic stability, we now introduce the notion of a resolvent matrix.

Definition 3.1 (*[11,13]*). The resolvent marix $R(n)$ of Eq. (3.1) is defined as the unique solution of the matrix equation

$$R(n + 1) = AR(n) + \sum_{j=0}^{n} B(n - j)R(j), \tag{3.4}$$

$R(0) = I, n \in \mathbb{Z}^+$.

Take the Z-transform of Eq. (3.4) yields

$$\tilde{R}(z) = zG^{-1}(z), \quad |z| > \mu. \tag{3.5}$$

The resolvent matrix $R(n)$ will be used to find the solution of the perturbed system

$$y(n + 1) = Ay(n) + \sum_{j=0}^{n} B(n - j)y(j) + g(n). \tag{3.6}$$

Taking the Z-transform of Eq. (3.6) yields

$$\begin{aligned} \tilde{y}(z) &= G^{-1}(z)[zy(0) + \tilde{g}(z)], \quad |z| > \mu \\ &= \tilde{R}(z)y(0) + \frac{1}{z}\tilde{R}(z)\tilde{g}(z), \quad |z| > \mu. \end{aligned}$$

Taking the inverse Z-transform we obtain

$$\boxed{y(n) = R(n)y_0 + \sum_{j=0}^{n-1} R(n - r - 1)g(r)}\,[2] \tag{3.7}$$

Formula (3.7) is called the variation of constants formula of Eq. (3.1).

We now return to our main focus, asymptotic stability. Next we state a fundamental result.

Let

$$h(n) := \sum_{r=0}^{\infty} \left| \sum_{j=0}^{n-1} R(n - j - 1)B(j + r + 1) \right|.$$

[2] $z[x(n - 1)] = \frac{1}{z}\tilde{x}(z)$.

Theorem 3.2 ([15]). *For Eq. (3.1), the following statements are equivalent.*

(a) $\det(zI - A - \tilde{B}(z)) \neq 0$, *for* $|z| \geq 1$
(b) $R(n) \in \ell^1(\mathbb{Z}^+)$
(c) *The zero solution of Eq. (3.1) is UAS*
(d) *Both* $R(n)$ *and* $h(n)$ *tend to zero as* $n \to \infty$

Proof. (a) ⇒ (b) Define the matrix sequence $\hat{B}(n) = B(n)$ if $n \neq 0$, $\hat{B}(0) = B(0) + A$. Then Eq. (3.4) may be written as

$$R(n+1) = \hat{B}(n) + \sum_{j=1}^{n} \hat{B}(n-j)R(j)$$

$$|R(n+1)| \leq \alpha + \sum_{j=1}^{n} |\hat{B}(n-j)||R(j)|.$$

By the discrete Gronwalls' inequality

$$|R(n)| \leq (1+\alpha)^n = \beta^n, \quad \alpha = \|B(n)\|.$$

Hence

$$\begin{aligned} \tilde{R}(z) &= z(zI - A - \tilde{B}(z))^{-1}, \quad |z| > \beta > 1 \\ &= \left(I - \frac{1}{z}A - \frac{1}{z}\tilde{B}(z)\right)^{-1}, \quad |z| > \beta > 1. \end{aligned}$$

For sufficiently large η,

$$\inf_{|z| > \eta} \left|\det\left(I - \frac{1}{z}A - \frac{1}{z}\tilde{B}(z)\right)\right| \geq \frac{1}{2}$$

Furthermore, on the compact annulus $1 \leq |z| \leq \eta$, $\inf \det\left(I - \frac{1}{z}A - \frac{1}{z}\tilde{B}(z)\right) \neq 0$. Consequently,

$$\inf \left|\det\left(I - \frac{1}{z}A - \frac{1}{z}\tilde{B}(z)\right)\right| > 0, \quad \text{for } |z| \geq 1.$$

By a Theorem of Wiener, there exists $H(n) \in \ell^1(\mathbb{Z}^+)$ such that

$$\tilde{H}(z)\left(I - \frac{1}{z}A - \frac{1}{z}\tilde{B}(z)\right) = I, \quad \text{for } |z| \geq 1.$$

By the uniqueness of the inverse,

$$\tilde{H}(z) = \tilde{R}(z) \in \ell^1,$$

and the proof is now complete.

(b) ⇒ (c) Assume that $R(n) \in \ell^1$. Then

$$x(n+\tau+1, \tau, \varphi) = A(x(n+\tau, \tau, \varphi)) + \sum_{j=0}^{n+\tau} B(n+\tau-j)x(j, \tau, \varphi),$$

where $\varphi : [0, s] \to \mathbb{R}^k$ is a given initial function, where $x(n) = \varphi(n)$ on $[0, s]$.

$$x(n+\tau+1, \tau, \varphi) = Ax(n+\tau, \tau, \varphi) + \sum_{j=0}^{n} B(n-j)x(j+\tau, \tau, \varphi) + \sum_{j=1}^{\tau} B(n+j)\varphi(\tau-j).$$

By the Variation of Constants formula

$$x(n+\tau, \tau, \varphi) = R(n)\varphi(\tau) + \sum_{j=0}^{n-1} R(n-j-1) \sum_{j=0}^{\tau} B(j+s)\varphi(\tau-s) \tag{3.8}$$

$$|x(n+\tau, \tau, \varphi)| \leq \|\varphi\|_{[0,\tau]}\left[|R(n)| + \sum_{j=0}^{n-1} |R(n-j-1)| \sum_{s=j+1}^{\infty} |B(s)|\right]. \tag{3.9}$$

Since $|R(n)| \to 0$ as $n \to \infty$, the second term in (3.9) tends to zero as it is the convolution of ℓ^1 sequence with a null sequence, the right hand side of (3.9) is bounded and tends to zero as $n \to \infty$. Hence the zero solution of Eq. (1.1) is *UAS*.

The proof of (c) ⇒ (d) and (d) ⇒ (a) will not be provided. □

Using Theorem 3.2, we give sufficient conditions for asymptotic stability. We also provide a partial converse. Let $\nu_{ij} = \sum_{n=0}^{\infty} b_{ij}(n) < \infty$.

Theorem 3.3 ([12]). *Let $A = (a_{ij})$ and $B(n) = (b_{ij}(n))$ such that*

$$\beta_{ij} = \sum_{n=0}^{\infty} |b_{ij}(n)| < \infty.$$

Then the zero solution of Eq. (3.1) is uniformly asymptotically stable if either one of the following conditions hold.

$$\text{(a)} \quad \sum_{i=1}^{k} (|a_{ij}| + \beta_{ij}) < 1, \quad 1 \le i \le k, \tag{3.10}$$

$$\text{(b)} \quad \sum_{i=1}^{k} (|a_{ij}| + \beta_{ij}) < 1, \quad 1 \le j \le k. \tag{3.11}$$

Theorem 3.4 ([12]). *Suppose that the following statements hold:*

1. $a_{ii} + \nu_{ii} > 1$, $1 \le i \le k$,
2. $(a_{ii} + \nu_{ii} - 1)(a_{jj} + \nu_{jj} - 1) > \sum_r' |a_{ir} + \nu_{ir}| \sum_r' |a_{jr} + \nu_{jr}|$, *where*

$$\sum_r{}' a_{ir} = \sum_{r=1}^{k} a_{ir} - a_{ii}.$$

Then if k is odd, the zero solution of Eq. (3.1) is not asymptotically stable. If k is even, then the zero solution of Eq. (3.1) may or may not be asymptotically stable.

4. Uniform asymptotic stability versus exponential stability

We commence this section with the following illustrative example.

Example 4.1. Consider the scalar equation

$$x(n+1) = \frac{1}{4}x(n) + \sum_{j=0}^{n} \frac{x(j)}{(2(n-j)+1)(2(n-j)+3)}.$$

Here $a = \frac{1}{4}$, $b(n) = 1/[(2n+1)(2n+3)]$ which is an ℓ^1 sequence. Since $a + \sum_{n=0}^{\infty} b(n) \le \frac{1}{4} + \frac{1}{2} < 1$, it follows that the zero solution is *UAS*. This raises the question of whether or not the zero solution is exponentially stable.

The next result provides the definitive answer to this question and shows that the zero solution is not exponenetially stable.

Theorem 4.2 ([15]). *Suppose that the zero solution of Eq (3.1) is UAS. Then the zero solution of Eq. (3.1) is exponentially stable if and only if $B(n)$ decays exponentially.*

5. Equations of nonconvolution type

In this section we consider the following system of Volterra difference equations of nonconvolution type

$$x(n+1) = A(n)x(n) + \sum_{j=0}^{n} B(n,j)x(j) \tag{5.1}$$

$$y(n+1) = A(n)y(n) + \sum_{j=0}^{n} B(n,j)y(j) + g(n), \tag{5.2}$$

where $A(n) = (a_{ij}(n))$, $B(n,m) = (b_{ij}(n,m))$ are $k \times k$ matrices on $\mathbb{Z}^+$, $\mathbb{Z}^+ \times \mathbb{Z}^+$, respectively, and $g(n)$ is a vector sequence on $\mathbb{Z}^+$.

Definition 5.1 ([11]). The resolvent matrix $R(n,m)$ of Eq. (5.1) is defined as the unique solution of the matrix difference equation

$$R(n+1,m) = A(n)R(n,m) + \sum_{j=m}^{n} B(n,m)R(j,m), \quad n \ge m, \tag{5.3}$$

with $R(m,m) = I$.

A variation of the constants formula

$$y(n, n_0, y_0) = R(n, n_0)y_0 + \sum_{j=0}^{n-1} R(n, j+1)g(j) \tag{5.4}$$

is the unique solution of Eq. (5.2) with $y(n_0) = y_0$.

The main disadvantage of dealing with equations of nonconvolution type is that we are unable to use the Z-transform methods and theory. Hence we are forced to use the method of Liapunov functions which is definitley much harder to construct.

Let

$$\beta_{ij}(n) = \sum_{s=n}^{\infty} |b_{ij}(s, n)|$$

and

$$\delta = \sup \sum_{r=0}^{n} \sum_{s=n}^{\infty} |b_{ij}(s, r)|.$$

Theorem 5.2 ([11]). *Assume that $\beta_{ij}(n) < \infty$ and $\delta < \infty$ and such that for $1 \le i \le k$, $n \ge 0$,*

$$\sum_{j=1}^{k} |a_{ji}(n)| + \beta_{ji}(n) \le 1 - \delta$$

for some $\delta \in (0, 1)$. Then the zero solution of Eq. (5.1) is globally UAS and in fact globally exponentially stable.

Proof. Let

$$V(n, x(\cdot)) = \sum_{j=1}^{k} \left[|x_i(n)| + \sum_{j=1}^{k} \sum_{r=0}^{n-1} \sum_{s=n}^{\infty} |b_{ij}(s, r)| \cdot |x_j(r)| \right].$$

Then it is easy to show that

$$\Delta V(n, x(\cdot)) \le -\delta V(n, x(\cdot)).$$

Hence

$$\begin{aligned} |x(n)| \le V(n, x(\cdot)) &\le (1-\delta)^n V(n_0, \varphi(\cdot)) \\ &\le M(1-\delta)^n \|\varphi\| \end{aligned}$$

where $\|\varphi\| = \sup\{|\varphi(s)| : s \in [0, n_0]\}$ is a bounded initial function.

A second approach to study stability is through the use of vector Liapunov functions. To simplify our notation, let us rewrite Eq. (5.1) in the form

$$x(n+1) = \sum_{j=0}^{n} C(n, j)x(j), \tag{5.5}$$

where $c(n, n) = A(n) + B(n, n)$ and $C(n, j) = B(n, j)$ for $n \ne j$. We define the absolute value of a matrix $A = (a_{ij})$ as the matrix $|A| = (|a_{ij}|)$. We say that $A \le B$ if $a_{ij} \le B_{ij}$, for $1 \le i, j \le k$. $\square$

Theorem 5.3 ([14]). *Suppose that for each $n \in \mathbb{Z}^+$, $\sum_{i=0}^{\infty} |C(i, n)| < \infty$ and the eigenvalues of the matrix $G = \sup_{n \ge 0} \{\sum_{i=n}^{\infty} |C(i, n)|\}$ lie inside the unit disc. Then the zero solution of (5.1) is UAS.*

Sketch of the Proof. Use the vector Liapunov functional

$$V(n, x(\cdot)) = (I - G)^{-1} \left[|x(n)| + \sum_{r=0}^{n-1} \sum_{s=n}^{\infty} |C(s, r)||x(r)| \right]. \quad \square$$

6. Asymptotic equivalence for difference equations with infinite delay

Consider the Volterra equations

$$x(n+1) = \sum_{s=-\infty}^{n} K(n-s)x(s), \quad n \geq n_0 \geq 0 \tag{6.1}$$

$$y(n+1) = \sum_{s=-\infty}^{n} \{K(n-s) + D(n,s)\}y(s), \quad n \geq n_0 \geq 0. \tag{6.2}$$

Assume

(H1) $\sum_{n=0}^{\infty} |K(n)|e^{\gamma n} < \infty$ and $\sum_{s=-\infty}^{n} \sup_{n\geq n_0} |D(n,s)|e^{\gamma(n-s)} < \infty$ for some $\gamma > 0$.

By virtue of Assumption (H1), Systems (6.1) and (6.2) are viewed as functional difference equations on the Banach space

$$B^\gamma = \left\{\varphi : \mathbb{Z}^- \to C^k : \sup_{t\in\mathbb{Z}^-} |\varphi(t)|e^{\gamma t} < \infty\right\}$$

equipped with the norm

$$\|\varphi\| = \sup_{t\in\mathbb{Z}^-} |\varphi(t)|e^{\gamma t} < \infty, \ \ \varphi \in B^\gamma,$$

where $\mathbb{Z}^- = \{\dots, -2, -1, 0\}$ and C denotes the complex plane. Indeed, System (6.1) can be written as a functional difference equation of the form

$$x(n+1) = L(x_n), \tag{6.3}$$

where $L(\cdot) : B^\gamma \to C^k$ is a functional defined by

$$L(\varphi) = \sum_{j=0}^{\infty} K(j)\varphi(-j), \ \ \varphi \in B^\gamma,$$

and x_n is a function in B^γ defined as

$$x_n(s) = x(n+s), \quad s \in \mathbb{Z}^-.$$

Let $T(n)$ denote the solution operator of Eq. (6.3). Then $T(n)\varphi = x_n(\varphi)$, for $\varphi \in B^\gamma$. Moreover, we will denote by $x(\cdot,\varphi)$ the solution of Eq. (6.3) satisfying $x(s,\varphi) = \varphi(s)$, for $s \in \mathbb{Z}^-$. It can be easily verified that $T(n)$ is a bounded linear operator on B^γ and satisfies the semigroup property

$$T(n+m) = T(n)T(m), \quad n, m \in \mathbb{Z}^+. \tag{6.4}$$

In the first two results we assume that Eq. (6.3) possesses an ordinary dichotomy. For the convenience of the reader, we now give its definition.

Let P be a projection on B^γ. Then B^γ can be written as a direct sum $B^\gamma = S \oplus U$, where S and U are closed subspaces of B^γ such that P is a projection from B^γ to S.

Definition 6.1 (*[16]*). System (6.3) is said to possess an ordinary dichotomy if there exists a projection P and a positive constant M such that

(i) S and U are invariant for $T(n)$,
(ii) $\|T(n)P\| \leq M$, for $n \in \mathbb{Z}^+$, and
(iii) $T(n)$ is extendable for $n \in \mathbb{Z}^-$ on U as a group with

$$\|T(n)(I-P)\| \leq M, \quad \text{for } n \in \mathbb{Z}^-.$$

In the sequel, M is referred to as the dichotomy constant.

Set

$$E^0(t) = \begin{cases} I, \ \text{the } k\times k \text{ identity matrix} & \text{if } t = 0,\\ 0, \ \text{the zero } k\times k \text{ matrix} & \text{if } t \neq 0.\end{cases}$$

To this end we have presented all the necessary preliminaries and groundwork. Hence, without further delay, we now state our main results.

Theorem 6.2 (*[16]*). *Suppose that Eq.* (6.3) *possesses an ordinary dichotomy and Assumption* (H1) *holds. Moreover, suppose that condition*

(H2) $\sum_{s=n_0}^{\infty} \sum_{j=-\infty}^{n_0-1} |D(s,j)|e^{\gamma(n_0-j)} + \sum_{s=n_0}^{\infty} \sum_{j=n_0}^{s} |D(s,j)| < 1/M,$

where M is the dichotomy constant, is satisfied. Then, for any bounded solution $x(n)$ of (6.1) on $[n_0,\infty)$ there exists a unique bounded solution $y(n)$ of (6.2) on $[n_0,\infty)$ such that

$$\begin{aligned} y_n &= x_n + \sum_{s=n_0}^{n-1} T(n-s-1)PE^0\left(\sum_{j=-\infty}^{s} D(s,j)y(j)\right) \\ &\quad - \sum_{s=n}^{\infty} T(n-s-1)(I-P)E^0\left(\sum_{j=-\infty}^{s} D(s,j)y(j)\right), \quad n \ge n_0. \end{aligned} \tag{6.5}$$

Conversely, for any bounded solution $y(n)$ of (6.2) on $[n_0,\infty)$ there exists a bounded solution $x(n)$ of (6.1) on $[n_0,\infty)$ satisfying the relation (6.5).

Theorem 6.3 (*[16]*). *Assume* (H1), (H2) *an ordinary dichotomy with the strenthened estimate*

$$\|T(n)P\| \le Ma^n \ (n \ge 0) \quad \text{for some } A \text{ with } 0 < a < 1. \tag{6.6}$$

Then there is a one to one correspondence between bounded solutions $x(n)$ of (6.1) on $[n_0,\infty)$ and bounded solutions $y(n)$ of (6.2) on $[n_0,\infty)$, and the asymptotic relation

$$y(n) = x(n) + o(1) \ (n \to \infty) \tag{6.7}$$

holds.

Theorem 6.4 (*[16]*). *Suppose that* (H1) *and the following two conditions are satisfied:*

(H3) $\sum_{s=n_0}^{\infty}\sum_{j=-\infty}^{s} |D(s,j)|e^{\gamma(s-j)} < \infty$;

(H4) *the roots of the equation*

$$\det\left(zI - \sum_{n=0}^{\infty} K(n)z^{-n}\right) = 0$$

are simple on the complex unit circle.

Then there is a one to one correspondence between bounded solutions $x(n)$ of (6.1) on $[n_1,\infty)$ and bounded solutions $y(n)$ of (6.2) on $[n_1,\infty)$, and the asymptotic relations (6.7) holds; here n_1 is a sufficienlty large integer.

It follows that $\det\left(zI - \sum_{n=0}^{\infty} K(n)z^{-n}\right) \ne 0$ for all $|z| \ge 1$ if the $k\times k$ matrix $K(n) = (K_{ij}(n))$ satisfies the following condition:

(H5) $\max_{1\le i\le k}\sum_{j=1}^{k}\sum_{n=0}^{\infty}|K_{ij}(n)| < 1$ or $\max_{1\le j\le k}\sum_{j=1}^{k}\sum_{n=0}^{\infty}|K_{ij}(n)| < 1$.

Therefore the following result is a direct consequence of Theorem 6.4.

Corollary 6.5 (*[16]*). *Assume* (H1), (H3) *and* (H5). *Then, for a sufficiently large integer n_1 there is a one to one correspondence between bounded solutions $x(n)$ of (6.1) on $[n_1,\infty)$ and bounded solutions $y(n)$ of (6.2) on $[n_1,\infty)$, and the asymptotic relations (6.7) holds.*

Before concluding this section, we provide an example to illustrate the usefulness of our results.

We consider the following scalar difference equation:

$$x(n+1) = 2x(n) - \sum_{s=-\infty}^{n}\left(\frac{1}{2}\right)^{n-s} x(s), \tag{6.8}$$

which is a special case of Eq. (6.1) with $K(0) = 1$ and $K(n) = -(1/2)^n$ for $n \ge 1$. Condition (H4) is satisfied for (6.8), because "the characteristic equation" $\det\left(zI - \sum_{n=0}^{\infty} K(n)z^{-n}\right) = 0$ yields the equation $2z^2 - 3z + 2 = 0$ whose roots $(3 \pm \sqrt{7}i)/4$ are simple.

Consider the perturbed equation

$$y(n+1) = 2y(n) - \sum_{s=-\infty}^{n}\left(\frac{1}{2}\right)^{n-s} y(s) + d(n)\sum_{s=-\infty}^{n} B(n-s)y(s), \tag{6.9}$$

where $\sum^{\infty}|d(n)| < \infty$ and $|B(n)| \le (1/2)^n$ for $n \in \mathbb{Z}^+$. Clearly, Conditions (H1) and (H3) are satisfied with $\gamma = \log(3/2)$. We note that $\tilde{x}(n) := ((3+\sqrt{7}i)/4)^n$ is a bounded solution of (6.8). By applying Theorem 6.4, we see that there exists a bounded solution which approaches to $\tilde{x}(n)$ as $n \to \infty$. We emphasize that Condition (H3) cannot necessarily be replaced by a weaker condition

$$\text{“}\sup_{s\ge n_0}\sum_{\tau=-\infty}^{s} |D(s,\tau)|e^{\gamma(s-r)} < \infty\text{”}$$

in Theorem 6.4. Indeed, when $d(n) \equiv d(-7/4 < d < 0)$, $B(0) = 1$ and $B(n) = -(1/2)^n$ for $n \ge 1$, any solution of (6.9) tends to zero as $n \to \infty$, because the characteristic equation of (6.9) is the equation $2z^2 - (3+2d)z + 2(1+d) = 0$ whose roots belong to the open unit disk in the complex plane. Therefore, no solutions of (6.9) can approach to the bounded solution $\tilde{x}(n)$ of (6.8) as $n \to \infty$, becuase of $|\tilde{x}(n)| \equiv 1$.

7. Open problems

Open Problem 1 Determine the stability of Eq. (1.1) when

$$A + \sum_{n=0}^{\infty} B(n) = -1 \quad \text{and} \quad \sum_{n=0}^{\infty} B(n) < 0.$$

Open Problem 2 Determine the stability of the zero solution of Eq. (1.1) when

$$-1 < A + \sum_{n=0}^{\infty} B(n) < 1.$$

Open Problem 3 If in Theorem 3.4 $a_{ii} + v_{ii} < 1$, for $1 \leq i \leq k$, what can we conclude about the stability of the zero solution of Eq. (1.1)?

Open Problem 4 Suppose that any one of the conditions in Theorem 3.2 holds. Then by Theorem 4.2 the zero solution of Eq. (3.1) is exponentially stable if and only if $B(n)$ is of exponential decay. Find an estimate of the rate of decay of solutions of Eq. (3.1) if $B(n)$ is not of exponential decay.

Consider Eqs. (5.1) and (5.2) with $A(n)$, $B(n, j)$, $g(n)$ almost periodic for $n, j \in \mathbb{Z}^+$ and $n \geq j$.

Open Problem 5 Find conditions under which Eq. (5.2) has an almost periodic solution.

Open Problem 6 Find conditions under which Eq. (5.2) has a unique asymptotically stable almost periodic solution.

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